Functions of Several Variables

Sketching

- Level Curves
- Sections

Partial Derivatives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

on every open set on which f and the partials,

$$\frac{\partial f}{x}, \frac{\partial f}{y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$$

are continuous.

Normal Vector

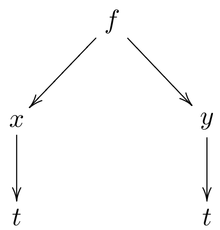
$$\mathbf{n} = \pm \begin{pmatrix} \frac{\partial f(x_0, y_0)}{\partial x} \\ \frac{\partial f(x_0, y_0)}{\partial y} \\ -1 \end{pmatrix}$$

Error Estimation

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right] \Delta x + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] \Delta y$$

Chain Rules

Example, $z = f(x, y)$ with $x = x(t), y = y(t)$



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Integration

Integration by Parts

Integrate the chain rule,

$$u(x)v(x) = \int u'(x)v(x)\ dx + \int v'(x)u(x)\ dx$$

Integration of Trig Functions

For $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$ remember that,

$$\cos 2x = \cos^2 x - \sin^2 x$$

Integrals of the form $\int \cos^m x \sin^n x \, dx$, when m or n are odd, you can factorise using $\cos^2 x + \sin^x = 1$ and then using,

$$\int \sin^k x \cos x \, dx = \frac{1}{k+1} \sin^{k+1} x + C$$
$$\int \cos^k x \sin x \, dx = \frac{-1}{k+1} \cos^{k+1} x + C$$

Reduction Formulae

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Partial Fractions

Example, assume

$$\frac{2x-1}{(x+3)(x+2)^2} = \frac{A}{x+3} + \frac{Bx+C}{(x+2)^2}$$

Now multiply both sides by $(x+2)(x+3)$ and equate coefficients.

ODE's

Separable ODE

Separate then integrate.

Linear ODE

The ODE:

$$\frac{dy}{dx} + f(x)y = g(x)$$

has solution,

$$y(x) = \frac{1}{u(x)} \left[\int u(x)g(x) \, dx + C \right]$$

where,

$$u(x) := e^{\int f(x) \, dx}$$

Exact ODE

The ODE:

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

or as,

$$M(x,y)dx + N(x,y)dy = 0$$

is exact when,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Assume solution is of the form,

$$F(x,y) = c$$

with,

$$M = \frac{\partial F}{\partial x} \qquad N = \frac{\partial F}{\partial y}$$

Integrate to find two equations equal to $F(x,y)$, then compare and find solution from assumed form.

Second Order ODE's

The ODE:

$$ay'' + ay' + by = f(x)$$

For the homogeneous case ($f(x) = 0$)
the characteristic equation will be $a\lambda^2 + b\lambda + c = 0$
If the characteristic equation has,
Two Distinct Real roots, (replace the λ's with the solutions to the characteristic eqn.)

$$y = Ae^{\lambda x} + Be^{\lambda x}$$

Repeated Real roots,

$$y = Ae^{\lambda x} + Bxe^{\lambda x}$$

Complex roots,

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

where,

$$\lambda = \alpha \pm \beta i$$

For the For the homogeneous case,

$$y = y_h + y_p$$

$$y = \text{solution to homogeneous case} + \text{particular solution}$$

Guess something that is in the same form as the RHS.
If $f(x) = P(x) \cos ax$(or sin) a guess for y_p is $Q_1(x) \cos ax + Q_2(x) \sin ax$

Taylor Series

Taylor Polynomials

For a differentiable function f the Taylor polynomial of order n at $x = a$ is,

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's Theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Sequences

$$\lim_{x \to \infty} f(x) = L \implies \lim_{n \to \infty} a_n = L$$

essentially says that when evaluating limits functions and sequences are identical.
A sequence diverges when $\lim_{n \to \infty} a_n = \pm\infty$ or $\lim_{n \to \infty} a_n$ does not exist.

Infinite Series

Telscoping Series

Most of the terms cancel out.

n-th term test (shows divergence)

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is non-zero.

Integral Test

Draw a picture. Use when you can easily find the integral.

p- series

The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

Comparison Test

Compare to a p-series.

Limit form of Comparison Test

Look at $\lim_{n\to\infty} \frac{a_n}{b_n}$ where b_n is usually a p-series.
If $= c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
If $= 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
If $= \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

The series converges if $\rho < 1$,
the series diverges if $\rho > 1$ or ρ is infinite,
and the test is inconclusive if $\rho = 1$.

Alternating Series Test

The series,

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

converges if,

1. The u_n's are all > 0,
2. $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N, and
3. $u_n \to 0$.

Absolute Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Taylor Series

Taylor Polynomials consist of adding a finite number of things together, whereas Taylor Series is an infinite sum.
The Maclaurin series is the Taylor series at $x = 0$.

Power Series

More Calculus

Average Value of a Function

$$\frac{\int_a^b f(x)\ dx}{b-a}$$

Arc Length

$$\text{Arc length over } [a,b] = \int_a^b \sqrt{1+f'(x)^2}\ dx$$

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt \qquad \text{(parametric)}$$

Speed

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

Surface Area of Revolution

$$2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\ dx$$

$$2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\ dt \qquad \text{(parametric)}$$