

Vector Spaces

Vector Spaces

Vector Space - 10 rules to satisfy, including (Where V is a vector space.)

- Closure under addition. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$
- Existence of zero. $\mathbf{0} \in V$
- Closure under scalar multiplication. If $\mathbf{u} \in V$ then $\lambda\mathbf{u} \in V$, where $\lambda \in \mathbb{R}$

Subspaces

Subspace Theorem:

A subset S of a vectorspace V is a subspace if:

- S contains the zero vector.
- If $\mathbf{u}, \mathbf{v} \in S$ then $\mathbf{u} + \mathbf{v} \in S$ and $\lambda\mathbf{u} \in S$ for all scalars λ .

Column Space

The column space of a matrix A is defined as the span of the columns of A , written $\text{col}(A)$.

Linear Independence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors.

- If we can find scalars $\alpha_1 + \alpha_2 + \dots + \alpha_n$ not all zero such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

then we call S a linearly dependent set and that the vectors in S are linearly dependent.

- If the only solution of

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ then S is called a linearly independent set and that the vectors in S are linearly independent.

Basis

A set B of vectors in a vectorspace V is called a basis if

- B is linearly independent, and
- $V = \text{span}(B)$.

An orthonormal basis is formed where all the basis vectors are unit length and are mutually orthogonal (the dot product of any two is zero).

Dimension

The dimension of a vectorspace V , written $\text{dim}(V)$ is the number of basis vectors.

Linear Transformations

Linear Map

A function T which maps from a vectorspace V to a vectorspace W is said to be linear if, for all vectors $\mathbf{u}, \mathbf{v} \in V$ and for any scalar λ ,

i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$,

ii) $T(\lambda\mathbf{u}) = \lambda T(\mathbf{u})$.

The columns of the transformation matrix are simply the images of the standard basis vectors.

The Kernel

$$\text{im}(T) = \text{col}(A)$$

The kernel of a linear map $T : V \rightarrow W$, written $\ker(T)$, consists of the set of vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$.

If A is the matrix representation of a linear map T , then the kernel of A is the solution set of $A\mathbf{x} = \mathbf{0}$.

Rank-Nullity

The dimension of the image of a linear map T is called the rank of T , written $\text{rank}(T)$. (Maximum number of linearly independent columns of A)

The dimension of the kernel of a linear map T is called the nullity of T , written $\text{nullity}(T)$. (Number of parameters in the solution set of $A\mathbf{x} = \mathbf{0}$)

If T is a linear map from V to W then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Eigenvectors and Eigenvalues

If A is a square matrix, $\mathbf{v} \neq \mathbf{0}$ and λ is a scalar such that,

$$A\mathbf{v} = \lambda\mathbf{v}$$

then \mathbf{v} is an eigenvector of A with eigenvalue λ .

Eigenvalues: Set $\det(A - \lambda I) = 0$ and solve for λ .

Eigenvectors: For each value of λ find the kernel of $(A - \lambda I)$.

Diagonalisation

If A has n (independent) eigenvectors then put,

$$P = (\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n) \text{ (eigenvectors } \mathbf{v})$$

and

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \text{ (eigenvalues } \lambda)$$

so then

$$A^k = PD^kP^{-1}, \text{ for each non-negative integer } k.$$

Remember that when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Systems of Differential Equations

The system $\begin{cases} \frac{dx}{dt} = 4x + y \\ \frac{dy}{dt} = x + 4y \end{cases}$ can be written $\mathbf{x}'(t) = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \mathbf{x}(t)$ where $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

We can guess the solution to be $\mathbf{x}(t) = \alpha \mathbf{v} e^{\lambda t}$ (and add for all the eigenvalues). Where \mathbf{v} and λ are the eigenvectors and eigenvalues respectively.

Probability and Statistics

Probability

Two events A and B are mutually exclusive if $A \cap B = \emptyset$.

$$P(A^c) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Independence

Two events A and B are physically independent of each other if the probability that one of them occurs is not influenced by the occurrence or non occurrence of the other. These two events are statistically independent if,

$$P(A \cap B) = P(A).P(B).$$

Conditional Probability

Probability of A given B is given by,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes Rule

Discrete Random Variables

$$p_k = P(X = k) \quad (\{p_k\} \text{ is the probability distribution})$$

where, X is a discrete random variable, and $P(X = k)$ is the probability that $X = k$.

For $\{p_k\}$ to be a probability distribution,

i) $p_k \geq 0$ for $k = 0, 1, 2, \dots$

ii) $p_0 + p_1 + \dots = 1$

Mean and Variance

$E(X)$ denotes the mean or expected value of X .

$$E(X) = \sum_{\text{all } k} k p_k$$

$$\text{Var}(X) = E(X^2) - E(X)^2 \quad \text{where } E(X^2) = \sum_{\text{all } k} k^2 p_k$$

Binomial Distribution

If we perform a binomial experiment (i.e. 2 outcomes) n times, and each time there is a probability p of success then,

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad \text{for } 0 \leq k \leq n \text{ and } 0 \text{ otherwise.}$$

Geometric Distribution

$$P(X = k) = p(1 - p)^{k-1}, \quad k = 1, 2, \dots$$